11.7.1 Exercises

For a link to all of the additional resources available for this section, click OSttS Chapter 11 materials.

In Exercises 1 - 20, find a polar representation for the complex number z and then identify $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\arg(z)$ and $\mathrm{Arg}(z)$.

For help with these exercises, click the resource below:

- Writing complex numbers in polar form

1. $z = 9 + 9i$
2. $z = 5 + 5i\sqrt{3}$
3. $z = 6i$
4. $z = -3\sqrt{2} + 3i\sqrt{2}$
5. $z = -6\sqrt{3} + 6i$
6. $z = -2$
7. $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$
8. $z = -3 - 3i$
9. $z = -5i$
10. $z = 2\sqrt{2} - 2i\sqrt{2}$
11. $z = 6$
12. $z = i\sqrt[3]{7}$
13. $z = 3 + 4i$
14. $z = \sqrt{2} + i$
15. $z = -7 + 24i$
16. $z = -2 + 6i$
17. $z = -12 - 5i$
18. $z = -5 - 2i$
19. $z = 4 - 2i$
20. $z = 1 - 3i$

In Exercises 21 - 40, find the rectangular form of the given complex number. Use whatever identities are necessary to find the exact values.

21. $z = 6\mathrm{cis}(0)$
22. $z = 2\mathrm{cis}\left(\frac{\pi}{6}\right)$
23. $z = 7\sqrt{2}\mathrm{cis}\left(\frac{\pi}{4}\right)$
24. $z = 3\mathrm{cis}\left(\frac{\pi}{2}\right)$
25. $z = 4\mathrm{cis}\left(\frac{2\pi}{3}\right)$
26. $z = \sqrt{6}\mathrm{cis}\left(\frac{3\pi}{4}\right)$
27. $z = 9\mathrm{cis}\left(\pi\right)$
28. $z = 3\mathrm{cis}\left(\frac{4\pi}{3}\right)$
29. $z = 7\mathrm{cis}\left(-\frac{3\pi}{4}\right)$
30. $z = \sqrt{13}\mathrm{cis}\left(\frac{3\pi}{2}\right)$
31. $z = \frac{1}{2}\mathrm{cis}\left(\frac{7\pi}{4}\right)$
32. $z = 12\mathrm{cis}\left(-\frac{\pi}{3}\right)$
33. $z = 8\mathrm{cis}\left(\frac{\pi}{12}\right)$
34. $z = 2\mathrm{cis}\left(\frac{7\pi}{8}\right)$
35. $z = 5\mathrm{cis}\left(\arctan\left(\frac{4}{3}\right)\right)$
36. $z = \sqrt{10}\mathrm{cis}\left(\arctan\left(\frac{1}{3}\right)\right)$
37. $z = 15\mathrm{cis}\left(\arctan\left(-2\right)\right)$
38. $z = \sqrt{3}\left(\arctan\left(-\sqrt{2}\right)\right)$
39. $z = 50\mathrm{cis}\left(\pi - \arctan\left(\frac{7}{24}\right)\right)$
40. $z = \frac{1}{2}\mathrm{cis}\left(\pi + \arctan\left(\frac{5}{12}\right)\right)$

For Exercises 41 - 52, use

$$z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i \quad \text{and} \quad w = 3\sqrt{2} - 3i\sqrt{2}$$

to compute the quantity. Express your answers in polar form using the principal argument.
For help with these exercises, click one or more of the resources below:

- Finding products and quotients of complex numbers in polar form
- Finding powers of complex numbers in polar form (DeMoivre's Theorem)

41. zw
42. $\dfrac{z}{w}$
43. $\dfrac{w}{z}$
44. z^4
45. w^3
46. z^5w^2
47. z^3w^2
48. $\dfrac{z^2}{w}$
49. $\dfrac{w}{z^2}$
50. $\dfrac{z^3}{w^2}$
51. $\dfrac{w^2}{z^3}$
52. $\left(\dfrac{w}{z}\right)^6$

In Exercises 53 - 64, use DeMoivre's Theorem to find the indicated power of the given complex number. Express your final answers in rectangular form.
For help with these exercises, click the resource below:

- Finding powers of complex numbers in polar form (DeMoivre's Theorem)

53. $\left(-2+2i\sqrt{3}\right)^3$
54. $(-\sqrt{3}-i)^3$
55. $(-3+3i)^4$
56. $(\sqrt{3}+i)^4$
57. $\left(\dfrac{5}{2}+\dfrac{5}{2}i\right)^3$
58. $\left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^6$
59. $\left(\dfrac{3}{2}-\dfrac{3}{2}i\right)^3$
60. $\left(\dfrac{\sqrt{3}}{3}-\dfrac{1}{3}i\right)^4$
61. $\left(\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i\right)^4$
62. $(2+2i)^5$
63. $(\sqrt{3}-i)^5$
64. $(1-i)^8$

In Exercises 65 - 76, find the indicated complex roots. Express your answers in polar form and then convert them into rectangular form.
For help with these exercises, click the resource below:

- Finding roots of complex numbers

65. the two square roots of $z = 4i$
66. the two square roots of $z = -25i$

67. the two square roots of $z = 1 + i\sqrt{3}$

68. the two square roots of $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

69. the three cube roots of $z = 64$

70. the three cube roots of $z = -125$

71. the three cube roots of $z = i$

72. the three cube roots of $z = -8i$

73. the four fourth roots of $z = 16$

74. the four fourth roots of $z = -81$

75. the six sixth roots of $z = 64$

76. the six sixth roots of $z = -729$

77. Use the Sum and Difference Identities in Theorem 10.16 or the Half Angle Identities in Theorem 10.19 to express the three cube roots of $z = \sqrt{2} + i\sqrt{2}$ in rectangular form. (See Example 11.7.4, number 3.)

78. Use a calculator to approximate the five fifth roots of 1. (See Example 11.7.4, number 4.)

79. According to Theorem 3.16 in Section 3.4, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In Exercise 28 in Section 8.7, we showed you how to factor this polynomial into the product of two irreducible quadratic factors using a system of non-linear equations. Now that we can compute the complex fourth roots of -4 directly, we can simply apply the Complex Factorization Theorem, Theorem 3.14, to obtain the linear factorization $p(x) = (x - (1 + i))(x - (1 - i))(x - (-1 + i))(x - (-1 - i))$. By multiplying the first two factors together and then the second two factors together, thus pairing up the complex conjugate pairs of zeros Theorem 3.15 told us we'd get, we have that $p(x) = (x^2 - 2x + 2)(x^2 + 2x + 2)$. Use the 12 complex 12^{th} roots of 4096 to factor $p(x) = x^{12} - 4096$ into a product of linear and irreducible quadratic factors.

80. Complete the proof of Theorem 11.14 by showing that if $w \neq 0$ than $\left|\frac{1}{w}\right| = \frac{1}{|w|}$.

81. Recall from Section 3.4 that given a complex number $z = a+bi$ its complex conjugate, denoted $\overline{z}$, is given by $\overline{z} = a - bi$.

    (a) Prove that $|\overline{z}| = |z|$.
    (b) Prove that $|z| = \sqrt{z\overline{z}}$
    (c) Show that $\text{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\text{Im}(z) = \dfrac{z - \overline{z}}{2i}$
    (d) Show that if $\theta \in \arg(z)$ then $-\theta \in \arg(\overline{z})$. Interpret this result geometrically.
    (e) Is it always true that $\text{Arg}(\overline{z}) = -\text{Arg}(z)$?

82. Given any natural number $n \geq 2$, the n complex n^{th} roots of the number $z = 1$ are called the $\boldsymbol{n}^{\textbf{th}}$ **Roots of Unity**. In the following exercises, assume that n is a fixed, but arbitrary, natural number such that $n \geq 2$.

    (a) Show that $w = 1$ is an n^{th} root of unity.

(b) Show that if both w_j and w_k are n^{th} roots of unity then so is their product $w_j w_k$.

(c) Show that if w_j is an n^{th} root of unity then there exists another n^{th} root of unity $w_{j'}$ such that $w_j w_{j'} = 1$. Hint: If $w_j = \text{cis}(\theta)$ let $w_{j'} = \text{cis}(2\pi - \theta)$. You'll need to verify that $w_{j'} = \text{cis}(2\pi - \theta)$ is indeed an n^{th} root of unity.

83. Another way to express the polar form of a complex number is to use the exponential function. For real numbers t, Euler's Formula defines $e^{it} = \cos(t) + i\sin(t)$.

(a) Use Theorem 11.16 to show that $e^{ix}e^{iy} = e^{i(x+y)}$ for all real numbers x and y.

(b) Use Theorem 11.16 to show that $\left(e^{ix}\right)^n = e^{i(nx)}$ for any real number x and any natural number n.

(c) Use Theorem 11.16 to show that $\dfrac{e^{ix}}{e^{iy}} = e^{i(x-y)}$ for all real numbers x and y.

(d) If $z = r\text{cis}(\theta)$ is the polar form of z, show that $z = re^{it}$ where $\theta = t$ radians.

(e) Show that $e^{i\pi} + 1 = 0$. (This famous equation relates the five most important constants in all of Mathematics with the three most fundamental operations in Mathematics.)

(f) Show that $\cos(t) = \dfrac{e^{it} + e^{-it}}{2}$ and that $\sin(t) = \dfrac{e^{it} - e^{-it}}{2i}$ for all real numbers t.

Checkpoint Quiz 11.7

1. Let $z = 3 - 3i$.

(a) Plot z in the complex plane.

(b) Find $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, and $\text{Arg}(z)$

(c) Find the polar form of z.

(d) Use DeMoivre's Theorem to find z^4. Write your answer in rectangular form.

2. Find the cube roots of $z = -8i$. Write your answers in rectangular form.

For worked out solutions to this quiz, click the links below:

- Solution Part 1
- Solution Part 2

11.7.2 Answers

1. $z = 9 + 9i = 9\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$, $\text{Re}(z) = 9$, $\text{Im}(z) = 9$, $|z| = 9\sqrt{2}$
   $\arg(z) = \left\{\frac{\pi}{4} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{4}$.

2. $z = 5 + 5i\sqrt{3} = 10\text{cis}\left(\frac{\pi}{3}\right)$, $\text{Re}(z) = 5$, $\text{Im}(z) = 5\sqrt{3}$, $|z| = 10$
   $\arg(z) = \left\{\frac{\pi}{3} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{3}$.

3. $z = 6i = 6\text{cis}\left(\frac{\pi}{2}\right)$, $\text{Re}(z) = 0$, $\text{Im}(z) = 6$, $|z| = 6$
   $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{2}$.

4. $z = -3\sqrt{2} + 3i\sqrt{2} = 6\text{cis}\left(\frac{3\pi}{4}\right)$, $\text{Re}(z) = -3\sqrt{2}$, $\text{Im}(z) = 3\sqrt{2}$, $|z| = 6$
   $\arg(z) = \left\{\frac{3\pi}{4} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{3\pi}{4}$.

5. $z = -6\sqrt{3} + 6i = 12\text{cis}\left(\frac{5\pi}{6}\right)$, $\text{Re}(z) = -6\sqrt{3}$, $\text{Im}(z) = 6$, $|z| = 12$
   $\arg(z) = \left\{\frac{5\pi}{6} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{5\pi}{6}$.

6. $z = -2 = 2\text{cis}\left(\pi\right)$, $\text{Re}(z) = -2$, $\text{Im}(z) = 0$, $|z| = 2$
   $\arg(z) = \{(2k + 1)\pi \,|\, k \text{ is an integer}\}$ and $\text{Arg}(z) = \pi$.

7. $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i = \text{cis}\left(\frac{7\pi}{6}\right)$, $\text{Re}(z) = -\frac{\sqrt{3}}{2}$, $\text{Im}(z) = -\frac{1}{2}$, $|z| = 1$
   $\arg(z) = \left\{\frac{7\pi}{6} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{5\pi}{6}$.

8. $z = -3 - 3i = 3\sqrt{2}\text{cis}\left(\frac{5\pi}{4}\right)$, $\text{Re}(z) = -3$, $\text{Im}(z) = -3$, $|z| = 3\sqrt{2}$
   $\arg(z) = \left\{\frac{5\pi}{4} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{3\pi}{4}$.

9. $z = -5i = 5\text{cis}\left(\frac{3\pi}{2}\right)$, $\text{Re}(z) = 0$, $\text{Im}(z) = -5$, $|z| = 5$
   $\arg(z) = \left\{\frac{3\pi}{2} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{\pi}{2}$.

10. $z = 2\sqrt{2} - 2i\sqrt{2} = 4\text{cis}\left(\frac{7\pi}{4}\right)$, $\text{Re}(z) = 2\sqrt{2}$, $\text{Im}(z) = -2\sqrt{2}$, $|z| = 4$
    $\arg(z) = \left\{\frac{7\pi}{4} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = -\frac{\pi}{4}$.

11. $z = 6 = 6\text{cis}\left(0\right)$, $\text{Re}(z) = 6$, $\text{Im}(z) = 0$, $|z| = 6$
    $\arg(z) = \{2\pi k \,|\, k \text{ is an integer}\}$ and $\text{Arg}(z) = 0$.

12. $z = i\sqrt[3]{7} = \sqrt[3]{7}\text{cis}\left(\frac{\pi}{2}\right)$, $\text{Re}(z) = 0$, $\text{Im}(z) = \sqrt[3]{7}$, $|z| = \sqrt[3]{7}$
    $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \frac{\pi}{2}$.

13. $z = 3 + 4i = 5\text{cis}\left(\arctan\left(\frac{4}{3}\right)\right)$, $\text{Re}(z) = 3$, $\text{Im}(z) = 4$, $|z| = 5$
    $\arg(z) = \left\{\arctan\left(\frac{4}{3}\right) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{4}{3}\right)$.

14. $z = \sqrt{2} + i = \sqrt{3}\text{cis}\left(\arctan\left(\frac{\sqrt{2}}{2}\right)\right)$, $\text{Re}(z) = \sqrt{2}$, $\text{Im}(z) = 1$, $|z| = \sqrt{3}$
$\arg(z) = \left\{\arctan\left(\frac{\sqrt{2}}{2}\right) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{\sqrt{2}}{2}\right)$.

15. $z = -7 + 24i = 25\text{cis}\left(\pi - \arctan\left(\frac{24}{7}\right)\right)$, $\text{Re}(z) = -7$, $\text{Im}(z) = 24$, $|z| = 25$
$\arg(z) = \left\{\pi - \arctan\left(\frac{24}{7}\right) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \pi - \arctan\left(\frac{24}{7}\right)$.

16. $z = -2 + 6i = 2\sqrt{10}\text{cis}\left(\pi - \arctan(3)\right)$, $\text{Re}(z) = -2$, $\text{Im}(z) = 6$, $|z| = 2\sqrt{10}$
$\arg(z) = \left\{\pi - \arctan(3) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \pi - \arctan(3)$.

17. $z = -12 - 5i = 13\text{cis}\left(\pi + \arctan\left(\frac{5}{12}\right)\right)$, $\text{Re}(z) = -12$, $\text{Im}(z) = -5$, $|z| = 13$
$\arg(z) = \left\{\pi + \arctan\left(\frac{5}{12}\right) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{5}{12}\right) - \pi$.

18. $z = -5 - 2i = \sqrt{29}\text{cis}\left(\pi + \arctan\left(\frac{2}{5}\right)\right)$, $\text{Re}(z) = -5$, $\text{Im}(z) = -2$, $|z| = \sqrt{29}$
$\arg(z) = \left\{\pi + \arctan\left(\frac{2}{5}\right) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(\frac{2}{5}\right) - \pi$.

19. $z = 4 - 2i = 2\sqrt{5}\text{cis}\left(\arctan\left(-\frac{1}{2}\right)\right)$, $\text{Re}(z) = 4$, $\text{Im}(z) = -2$, $|z| = 2\sqrt{5}$
$\arg(z) = \left\{\arctan\left(-\frac{1}{2}\right) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan\left(-\frac{1}{2}\right) = -\arctan\left(\frac{1}{2}\right)$.

20. $z = 1 - 3i = \sqrt{10}\text{cis}\left(\arctan(-3)\right)$, $\text{Re}(z) = 1$, $\text{Im}(z) = -3$, $|z| = \sqrt{10}$
$\arg(z) = \left\{\arctan(-3) + 2\pi k \,|\, k \text{ is an integer}\right\}$ and $\text{Arg}(z) = \arctan(-3) = -\arctan(3)$.

21. $z = 6\text{cis}(0) = 6$

22. $z = 2\text{cis}\left(\frac{\pi}{6}\right) = \sqrt{3} + i$

23. $z = 7\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) = 7 + 7i$

24. $z = 3\text{cis}\left(\frac{\pi}{2}\right) = 3i$

25. $z = 4\text{cis}\left(\frac{2\pi}{3}\right) = -2 + 2i\sqrt{3}$

26. $z = \sqrt{6}\text{cis}\left(\frac{3\pi}{4}\right) = -\sqrt{3} + i\sqrt{3}$

27. $z = 9\text{cis}(\pi) = -9$

28. $z = 3\text{cis}\left(\frac{4\pi}{3}\right) = -\frac{3}{2} - \frac{3i\sqrt{3}}{2}$

29. $z = 7\text{cis}\left(-\frac{3\pi}{4}\right) = -\frac{7\sqrt{2}}{2} - \frac{7\sqrt{2}}{2}i$

30. $z = \sqrt{13}\text{cis}\left(\frac{3\pi}{2}\right) = -i\sqrt{13}$

31. $z = \frac{1}{2}\text{cis}\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$

32. $z = 12\text{cis}\left(-\frac{\pi}{3}\right) = 6 - 6i\sqrt{3}$

33. $z = 8\text{cis}\left(\frac{\pi}{12}\right) = 4\sqrt{2+\sqrt{3}} + 4i\sqrt{2-\sqrt{3}}$

34. $z = 2\text{cis}\left(\frac{7\pi}{8}\right) = -\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}$

35. $z = 5\text{cis}\left(\arctan\left(\frac{4}{3}\right)\right) = 3 + 4i$

36. $z = \sqrt{10}\text{cis}\left(\arctan\left(\frac{1}{3}\right)\right) = 3 + i$

37. $z = 15\text{cis}\left(\arctan(-2)\right) = 3\sqrt{5} - 6i\sqrt{5}$

38. $z = \sqrt{3}\text{cis}\left(\arctan\left(-\sqrt{2}\right)\right) = 1 - i\sqrt{2}$

39. $z = 50\text{cis}\left(\pi - \arctan\left(\frac{7}{24}\right)\right) = -48 + 14i$

40. $z = \frac{1}{2}\text{cis}\left(\pi + \arctan\left(\frac{5}{12}\right)\right) = -\frac{6}{13} - \frac{5i}{26}$

In Exercises 41 - 52, we have that $z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i = 3\text{cis}\left(\frac{5\pi}{6}\right)$ and $w = 3\sqrt{2} - 3i\sqrt{2} = 6\text{cis}\left(-\frac{\pi}{4}\right)$ so we get the following.

41. $zw = 18\text{cis}\left(\frac{7\pi}{12}\right)$
42. $\frac{z}{w} = \frac{1}{2}\text{cis}\left(-\frac{11\pi}{12}\right)$
43. $\frac{w}{z} = 2\text{cis}\left(\frac{11\pi}{12}\right)$
44. $z^4 = 81\text{cis}\left(-\frac{2\pi}{3}\right)$
45. $w^3 = 216\text{cis}\left(-\frac{3\pi}{4}\right)$
46. $z^5w^2 = 8748\text{cis}\left(-\frac{\pi}{3}\right)$
47. $z^3w^2 = 972\text{cis}(0)$
48. $\frac{z^2}{w} = \frac{3}{2}\text{cis}\left(-\frac{\pi}{12}\right)$
49. $\frac{w}{z^2} = \frac{2}{3}\text{cis}\left(\frac{\pi}{12}\right)$
50. $\frac{z^3}{w^2} = \frac{3}{4}\text{cis}(\pi)$
51. $\frac{w^2}{z^3} = \frac{4}{3}\text{cis}(\pi)$
52. $\left(\frac{w}{z}\right)^6 = 64\text{cis}\left(-\frac{\pi}{2}\right)$
53. $\left(-2 + 2i\sqrt{3}\right)^3 = 64$
54. $(-\sqrt{3} - i)^3 = -8i$
55. $(-3 + 3i)^4 = -324$
56. $(\sqrt{3} + i)^4 = -8 + 8i\sqrt{3}$
57. $\left(\frac{5}{2} + \frac{5}{2}i\right)^3 = -\frac{125}{4} + \frac{125}{4}i$
58. $\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^6 = 1$
59. $\left(\frac{3}{2} - \frac{3}{2}i\right)^3 = -\frac{27}{4} - \frac{27}{4}i$
60. $\left(\frac{\sqrt{3}}{3} - \frac{1}{3}i\right)^4 = -\frac{8}{81} - \frac{8i\sqrt{3}}{81}$
61. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^4 = -1$
62. $(2 + 2i)^5 - 128 - 128i$
63. $(\sqrt{3} - i)^5 = -16\sqrt{3} - 16i$
64. $(1 - i)^8 = 16$
65. Since $z = 4i = 4\text{cis}\left(\frac{\pi}{2}\right)$ we have

 $w_0 = 2\text{cis}\left(\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2}$ $\qquad$ $w_1 = 2\text{cis}\left(\frac{5\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$

66. Since $z = -25i = 25\text{cis}\left(\frac{3\pi}{2}\right)$ we have

 $w_0 = 5\text{cis}\left(\frac{3\pi}{4}\right) = -\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$ $\qquad$ $w_1 = 5\text{cis}\left(\frac{7\pi}{4}\right) = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

67. Since $z = 1 + i\sqrt{3} = 2\text{cis}\left(\frac{\pi}{3}\right)$ we have

 $w_0 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$ $\qquad$ $w_1 = \sqrt{2}\text{cis}\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

68. Since $z = \frac{5}{2} - \frac{5\sqrt{3}}{2}i = 5\text{cis}\left(\frac{5\pi}{3}\right)$ we have

 $w_0 = \sqrt{5}\text{cis}\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{15}}{2} + \frac{\sqrt{5}}{2}i$ $\qquad$ $w_1 = \sqrt{5}\text{cis}\left(\frac{11\pi}{6}\right) = \frac{\sqrt{15}}{2} - \frac{\sqrt{5}}{2}i$

69. Since $z = 64 = 64\text{cis}(0)$ we have

 $w_0 = 4\text{cis}(0) = 4$ $\qquad$ $w_1 = 4\text{cis}\left(\frac{2\pi}{3}\right) = -2 + 2i\sqrt{3}$ $\qquad$ $w_2 = 4\text{cis}\left(\frac{4\pi}{3}\right) = -2 - 2i\sqrt{3}$

70. Since $z = -125 = 125\mathrm{cis}\left(\pi\right)$ we have

$w_0 = 5\mathrm{cis}\left(\frac{\pi}{3}\right) = \frac{5}{2} + \frac{5\sqrt{3}}{2}i$ $\quad w_1 = 5\mathrm{cis}\left(\pi\right) = -5$ $\quad w_2 = 5\mathrm{cis}\left(\frac{5\pi}{3}\right) = \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

71. Since $z = i = \mathrm{cis}\left(\frac{\pi}{2}\right)$ we have

$w_0 = \mathrm{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\quad w_1 = \mathrm{cis}\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\quad w_2 = \mathrm{cis}\left(\frac{3\pi}{2}\right) = -i$

72. Since $z = -8i = 8\mathrm{cis}\left(\frac{3\pi}{2}\right)$ we have

$w_0 = 2\mathrm{cis}\left(\frac{\pi}{2}\right) = 2i$ $\quad w_1 = 2\mathrm{cis}\left(\frac{7\pi}{6}\right) = -\sqrt{3} - i$ $\quad w_2 = \mathrm{cis}\left(\frac{11\pi}{6}\right) = \sqrt{3} - i$

73. Since $z = 16 = 16\mathrm{cis}\left(0\right)$ we have

$w_0 = 2\mathrm{cis}\left(0\right) = 2$ $\quad w_1 = 2\mathrm{cis}\left(\frac{\pi}{2}\right) = 2i$

$w_2 = 2\mathrm{cis}\left(\pi\right) = -2$ $\quad w_3 = 2\mathrm{cis}\left(\frac{3\pi}{2}\right) = -2i$

74. Since $z = -81 = 81\mathrm{cis}\left(\pi\right)$ we have

$w_0 = 3\mathrm{cis}\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$ $\quad w_1 = 3\mathrm{cis}\left(\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$w_2 = 3\mathrm{cis}\left(\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$ $\quad w_3 = 3\mathrm{cis}\left(\frac{7\pi}{4}\right) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$

75. Since $z = 64 = 64\mathrm{cis}(0)$ we have

$w_0 = 2\mathrm{cis}(0) = 2$ $\quad w_1 = 2\mathrm{cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$ $\quad w_2 = 2\mathrm{cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$

$w_3 = 2\mathrm{cis}\left(\pi\right) = -2$ $\quad w_4 = 2\mathrm{cis}\left(-\frac{2\pi}{3}\right) = -1 - \sqrt{3}i$ $\quad w_5 = 2\mathrm{cis}\left(-\frac{\pi}{3}\right) = 1 - \sqrt{3}i$

76. Since $z = -729 = 729\mathrm{cis}(\pi)$ we have

$w_0 = 3\mathrm{cis}\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$ $\quad w_1 = 3\mathrm{cis}\left(\frac{\pi}{2}\right) = 3i$ $\quad w_2 = 3\mathrm{cis}\left(\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$w_3 = 3\mathrm{cis}\left(\frac{7\pi}{6}\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$ $\quad w_4 = 3\mathrm{cis}\left(-\frac{3\pi}{2}\right) = -3i$ $\quad w_5 = 3\mathrm{cis}\left(-\frac{11\pi}{6}\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

77. Note: In the answers for w_0 and w_2 the first rectangular form comes from applying the appropriate Sum or Difference Identity ($\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ and $\frac{17\pi}{12} = \frac{2\pi}{3} + \frac{3\pi}{4}$, respectively) and the second comes from using the Half-Angle Identities.

$w_0 = \sqrt[3]{2}\mathrm{cis}\left(\frac{\pi}{12}\right) = \sqrt[3]{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right) = \sqrt[3]{2}\left(\frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2}\right)$

$w_1 = \sqrt[3]{2}\mathrm{cis}\left(\frac{3\pi}{4}\right) = \sqrt[3]{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$w_2 = \sqrt[3]{2}\mathrm{cis}\left(\frac{17\pi}{12}\right) = \sqrt[3]{2}\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{-\sqrt{2}-\sqrt{6}}{4}\right)\right) = \sqrt[3]{2}\left(\frac{\sqrt{2-\sqrt{3}}}{2} + i\frac{\sqrt{2+\sqrt{3}}}{2}\right)$

78. $w_0 = \text{cis}(0) = 1$
$w_1 = \text{cis}\left(\frac{2\pi}{5}\right) \approx 0.309 + 0.951i$
$w_2 = \text{cis}\left(\frac{4\pi}{5}\right) \approx -0.809 + 0.588i$
$w_3 = \text{cis}\left(\frac{6\pi}{5}\right) \approx -0.809 - 0.588i$
$w_4 = \text{cis}\left(\frac{8\pi}{5}\right) \approx 0.309 - 0.951i$

79. $p(x) = x^{12} - 4096 = (x-2)(x+2)(x^2+4)(x^2-2x+4)(x^2+2x+4)(x^2-2\sqrt{3}x+4)(x^2+2\sqrt{3}+4)$